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On the breakage problem with a homogeneous erosion type kernel

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Abstract

The breakage equation with a homogeneous erosion type kernel is studied herein. This type of kernel renders the handling of the breakage equation by conventional techniques very difficult, necessitating alternative problem solving approaches. Exploiting the structure of the erosion-breakage kernel, a new particle erosion equation is derived as the first-order term of a formal perturbation expansion with respect to kernel parameters. However, even this new equation is very difficult to treat because of the multimodality of its solution associated with the developing generations of fragments. In order to overcome this difficulty, the problem is decomposed into a system of equations for the size distribution of the generations of fragments which admits unimodal solutions. The properties and the methods of solution (analytical, method of moments, etc) are studied extensively. Using solution techniques developed in this paper, results are reported for some simple cases, revealing a very interesting and rather unusual structure of the solutions of the erosion-breakage equation.

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1. Introduction

The phenomenon of breakage is very important in science and engineering. It is encountered in the literature under several names depending on the particular physical situation, for example crushing, milling, grinding, fracture, partition, disintegration, shattering, scission and fragmentation. The list of application areas is enormous. Examples in process technology include the combustion of coal (percolative fragmentation) [1], polymer processing (polymer degradation) [2], mineral processing (grinding) [3], biotechnology (breakage) [4], fluidized bed reactors (attrition of catalyst particles) [5], slurry flow [6] and crystallization (crystal breakage) [7]. Applications in other scientific fields are also numerous, ranging from meteorology (rain formation) [8] and astronomy (size distribution of asteroids) [9] to the car parking problem of statistical physics [10].

In all application areas there is a clear distinction between the two modes of the breakage process. When the breakage event produces a ‘coarse’ fragment of size close to the parent particle and a number of much smaller ‘fine’ particles, the process is called particle erosion [11], attrition [12], abrasion [13] or chain-end scission [14] depending on the physical situation. In any other case the process is usually referred to as fracture. Redner [15] further divides the fracture mode into cleavage (involving a small number of fragments of similar size) and destructive breakage or shattering (producing many fragments with a wide spectrum of sizes). The above modes of breakage are shown schematically in [13,15]. In many cases the two main modes (particle erosion and fracture) may occur simultaneously as separate processes [11,16].

The first mode of breakage (referred to as *erosion breakage* henceforth) is at least as important as fracture (to be called simply *breakage* hereafter) in many application areas. It must be pointed out here that the methods for solving the breakage equation are generally very efficient in the case of fracture, but quite inefficient in the erosion case. A fundamentally different approach is required for tackling the latter. In the simplest case of erosion breakage, that of fixed-size fragments (independent of the parent particle size), the breakage equation can be replaced by a first-order hyperbolic PDE [16], which is well known from the study of dissolution of solids in liquids [17] and aerosol evaporation [18]. The new equation can be derived in a way similar to that of the Fokker–Planck equation in statistical physics [19] and in many cases it is used (e.g. [20]) directly without reference to the breakage equation.

Another simple class of erosion-breakage processes is that of a fragment size directly proportional to the size of the parent particle (*homogeneous breakage kernel*). In the majority of studies of the breakage problem, a homogeneous kernel is assumed. It is, therefore, reasonable to select to study the erosion-breakage process with such a kernel. Furthermore, the solution of this problem has a much richer structure in comparison with the relevant problem of fixed fragment size, mainly treated in the literature so far [11, 16, 20].

The structure of this paper is as follows: section 2 includes an outline of the general breakage equation along with some solution techniques, an analysis of the properties of the general breakage kernel (with particular emphasis on the homogeneous erosion kernel) and the derivation of the erosion equation from the general breakage equation. In section 3 the new erosion equation is decomposed into a system of partial integrodifferential equations. The analytical solution of this system is presented and an approximate solution based on the method of moments with log-normal distribution is discussed in some detail. Finally, using the methods of solution described in section 3, typical results (analytical and numerical) for simple test cases (such as constant breakage rate and monodisperse initial distribution) are obtained and discussed in section 4.

2. Problem formulation

2.1. The breakage equation

The breakage process can be described in general by the following linear partial integrodifferential equation:

$$\frac{\partial f'(x',\tau)}{\partial \tau} = \int_{x'}^{\infty} p'(x',y')b'(y')f'(y',\tau)\,\mathrm{d}y' - b'(x')f'(x',\tau) \tag{1}$$

where τ is time, x' particle volume, $f'(x',\tau)$ particle number density distribution, $b'(x')$ breakage rate and $p'(x',y')$ the probability distribution of particles of volume x' resulting from the breakup of a particle of volume y'.

Let $f_0'(x') = f'(x',0)$ be the initial particle size distribution. The total volume concentration, the total number concentration and the mean size of the initial distribution are, respectively,

$$M = \int_0^\infty x f_0'(x)\,\mathrm{d}x \tag{2a}$$

$$N_0 = \int_0^\infty f_0'(x)\,\mathrm{d}x \tag{2b}$$

$$x_0 = \frac{M}{N_0}. \tag{2c}$$

The functions and variables already introduced can be expressed in dimensionless form as follows:

$$\begin{aligned} &x = \frac{x'}{x_0} \qquad y = \frac{y'}{x_0} \qquad t = b'(x_0)\tau \qquad f(x,t) = \frac{x_0 f'(x',t)}{N_0} \\ &b(x) = b'(x')/b'(x_0) \qquad p(x,y) = p'(x',y')/x_0 \end{aligned} \tag{3}$$

and equation (1) can be written as

$$\frac{\partial f(x,t)}{\partial t} = \int_x^\infty p(x,y)b(y)f(y,t)\,\mathrm{d}y - b(x)f(x,t). \tag{4}$$

This equation has been solved by Ziff and McGrady [2] for a binary breakage kernel assuming a power series with respect to time and substituting in equation (4) to obtain recursive relations for the series coefficients, which are functions of particle size x. The generalization of this method for a general kernel is straightforward. For a monodisperse initial distribution $f_0(x) = \delta(x-1)$ the solution of equation (4) for arbitrary breakage kernel and rate is

$$f_\delta(x,t) = \sum_{i=0}^{\infty} A^{(i)}(x)\frac{t^{i+1}\mathrm{e}^{-t}}{(i+1)!} + \delta(x-1)\mathrm{e}^{-t} \tag{5a}$$

where

$$\begin{aligned} &A^{(i+1)}(x) = \int_x^1 \frac{1}{y} p(x,y)b(y)A^{(i)}(y)\,\mathrm{d}y + (1-b(x))A^{(i)}(x) \\ &i = 0,1,2,\ldots,\infty \qquad \text{and} \qquad A^{(0)}(x) = p(x,1). \end{aligned} \tag{5b}$$

It will be pointed out that the series solution represented by equations (5a), (5b) for constant breakage rate is reported by Lensu [21] by using the Mellin (moment) transformation [22]. As regards particular cases for the breakage kernel, the one with a power-law breakage kernel and power-law breakage rate has received considerable attention; i.e. McGrady and Ziff [23] gave a series solution based on the same approach used for the general kernel whereas Huang *et al* [24] obtained directly the (equivalent) solution in terms of the confluent hypergeometric function using Laplace transform techniques.

Due to the linearity of the breakage equation, the superposition principle holds. This means that the solution for an arbitrary initial distribution can be obtained from the solution for the monodisperse initial distribution as

$$f(x,t) = \int_0^\infty f_0(z)\frac{1}{z} f_\delta(x/z, b(z)t)\,\mathrm{d}z \tag{6}$$

where when the breakage kernel and rate do not depend explicitly on parent particle size (homogeneous case) the function f_δ is independent of z and must be evaluated once. In general the function f_δ must be computed via an equation similar to (5a) for every z value. The usual approach for the application of the superposition principle is to solve for the dimensional monodisperse distribution $\delta(x-z)$ and to integrate over z [25]. The derivation of equation (6) is not so straightforward; nevertheless, it depicts better the fact that only one solution (that for initial distribution $\delta(x-1)$) is needed for the case of homogeneous breakage kernel and rate.

2.2. Properties of erosion-breakage kernels

In general the function $p(x, y)$ should satisfy the following requirements:

(i) Conservation of mass:

$$\int_0^y xp(x, y)\,\mathrm{d}x = y. \tag{7}$$

This equation stipulates that the total volume of particles resulting from the breakup of a particle of volume y must be equal to y. In some studies in the physics literature a more general condition is assumed by multiplying the right-hand side of equation (7) by $(1-\lambda)$ where $0 < \lambda < 1$ [26]. This means that in each breakage event a fraction λ of the mass of the parent particle disappears. This case will not be considered in this paper.

(ii)
$$\int_0^k xp(x, y)\,\mathrm{d}x \geqslant \int_{y-k}^y (y-x)p(x, y)\,\mathrm{d}x \qquad \text{where} \quad k < \frac{y}{2}. \tag{8}$$

This expression states the requirement that only breakage events take place with no rearrangement of mass allowed. It is an absolute condition based on the physical requirement that when breakage occurs such that a particle $x \geqslant y/2$ is formed, the volume contained within the smaller fragments $(y-x)$ must contribute to the total volume of the fragments smaller than $(y-x)$. Further analysis concerning this condition can be found elsewhere [23]. For binary breakage (two fragments per parent particle), the above restriction is simplified being equivalent to a symmetric kernel in the sense $p(x, y) = p(y-x, y)$.

The above restriction on the form of breakage kernel is very important but it seems to be overlooked in the literature of breakage; this leads to physically unrealistic kernels used to fit experimental data. For example in [27] a kernel is employed (power law kernel with positive exponent) which violates condition (ii) and does not have any physical significance, even if it is capable of fitting the experimental data. In that case a different breakage mechanism (and different breakage kernel) may be dominant.

(iii) The number of particles resulting from breakage of a single particle of volume y is given as

$$v(y) = \int_0^y p(x, y)\,\mathrm{d}x. \tag{9}$$

According to the homogeneous erosion-breakage kernel examined in this paper, it is assumed that the particle size is continuous in all scales and the size of the fragments is defined as a fraction of the parent particle size independent of the absolute value of the latter. In turn, the fragments suffer breakage according to the same homogeneous breakage law. The homogeneity of the kernel implies that it can be written in the form $p(x, y) = P(x/y)/y$. The above properties (i) and (iii) of the breakage kernel may be transformed, respectively, to

$$\int_0^1 zP(z)\,\mathrm{d}z = 1 \tag{10}$$

$$v = \int_0^1 P(z)\,\mathrm{d}z. \tag{11}$$

The erosion type kernel has the following general form:

$$\begin{aligned} P(z) &= P_1(z) && \text{for} \quad 0 < z < \varepsilon_1 \\ &= 0 && \text{for} \quad \varepsilon_1 < z < \varepsilon_2 \\ &= P_2(z) && \text{for} \quad 1-\varepsilon_2 < z < 1 \end{aligned} \tag{12}$$

with $\varepsilon_2 \ll 1$. Using the above property (ii) results in the restriction $\varepsilon_1 \leqslant \varepsilon_2$, where the equality sign is for the binary breakage case. The parent particles retain their identity (and total number) so

$$\int_0^{\varepsilon_2} P_2(1-z)\,\mathrm{d}z = 1 \tag{13a}$$

$$\int_0^{\varepsilon_1} P_1(z)\,\mathrm{d}z = v - 1. \tag{13b}$$

Substituting the kernel into the mass conservation equation (10), rearranging and using (13a) results in

$$\int_0^{\varepsilon_1} zP_1(z)\,\mathrm{d}z = \int_0^{\varepsilon_2} zP_2(1-z)\,\mathrm{d}z. \tag{14}$$

It must be noted that many kernels defined in the entire interval [0,1] can be cast in the form of the erosion kernel with a very good approximation. For example the family of the breakage kernels of sum form developed by Hill and Ng [28] are practically zero everywhere except at the edges of the interval [0,1], for high values of its exponent. This is also the case for the U-type kernel developed by Kostoglou *et al* [29] for some values of its parameters.

2.3. The erosion equation

The breakage equation (4) with the homogeneous erosion breakage kernel (12) takes the form

$$\frac{\partial f(x,t)}{\partial t} = \int_x^{x/(1-\varepsilon_2)} \frac{1}{y} P_2(x/y) b(y) f(y,t)\,\mathrm{d}y - b(x)f(x,t) + \int_{x/\varepsilon_1}^{\infty} \frac{1}{y} P_1(x/y) b(y) f(y,t)\,\mathrm{d}y. \tag{15}$$

The first integral is along a small region of sizes greater than x. A new integration variable s is used to denote the fractional deviation from x. The new variable is defined by the equation $y = (1+s)x$ so that $0 < s < \varepsilon_2/(1-\varepsilon_2)$. The Taylor series expansion around x of the product $b(y)f(y,t)$ appearing in the integral is as follows:

$$b(y)f(y,t) = b(x+sx)f(x+sx,t) = b(x)f(x,t) + \sum_{i=1}^{\infty} \frac{(sx)^i}{i!} \frac{\partial^i b(x)f(x,t)}{\partial x^i}. \tag{16}$$

After substitution of the above series, the first and second terms on the right-hand side of equation (15) take the form

$$(K_0 - 1)b(x)f(x,t) + \sum_{i=1}^{\infty} \frac{x^i K_i}{i!} \frac{\partial^i b(x)f(x,t)}{\partial x^i} \tag{17}$$

where

$$K_i = \int_0^{\varepsilon_2/(1-\varepsilon_2)} \frac{s^i}{1+s} P_2\left(\frac{1}{1+s}\right)\mathrm{d}s = \int_0^{\varepsilon_2} \frac{z^i}{(1-z)^{i+1}} P_2(1-z)\,\mathrm{d}z. \tag{18}$$

In order to obtain a formal expansion of the breakage equation, with respect to parameter ε_2, the following series expansion is used:

$$\frac{z^i}{(1-z)^{i+1}} = \sum_{j=i}^{\infty} a_{ij} z^j \tag{19}$$

where $a_{ii} = 1$ and $a_{ij} = \frac{1}{(j-i)!}\prod_{k=1}^{j-i}(i+k)$ for $j > i$.

Substitution of the above series in the expression for K_i results in

$$K_i = \sum_{j=i}^{\infty} a_{ij} n_j \varepsilon_2^j \tag{20}$$

where n_j are defined as

$$n_j = \varepsilon_2 \int_0^1 z^j P_2(1 - z\varepsilon_2)\, \mathrm{d}z \tag{21}$$

and represent some kind of dimensionless moments of the kernel P_2. Using the condition $n_0 = 1$ (from equation (13a)), which is an intrinsic property of the breakage kernel, it can be inferred that the n_j are not a function of ε_2 and the series in equation (20) is a formal perturbation expansion of the K_i for small ε_2.

The series form of K_i is substituted in the breakage equation, the summation order of the double sum term is reversed and the terms with the same order of ε_2 dependence are collected to give the perturbation expansion with respect to ε_2

$$\frac{\partial f(x,t)}{\partial t} = \sum_{j=1}^{\infty} n_j \varepsilon_2^j \left[a_{0j} b(x) f(x,t) + \sum_{i=1}^{j} a_{ij} \frac{x^i}{i!} \frac{\partial^i b(x) f(x,t)}{\partial x^i} \right] + \int_{x/\varepsilon_1}^{\infty} \frac{1}{y} P_1(x/y) b(y) f(y,t)\, \mathrm{d}y. \tag{22}$$

The terms in the brackets for $j = 1$ and 2 can be written in the form

$$j = 1 \qquad \frac{\partial x b(x) f(x,t)}{\partial x} \tag{23a}$$

$$j = 2 \qquad 2\frac{\partial x b(x) f(x,t)}{\partial x} - b(x) f(x,t) + \frac{x^2}{2} \frac{\partial^2 b(x) f(x,t)}{\partial x^2}. \tag{23b}$$

For moderate values of ε_2, more than one term of the expansion must be used, in which case the $j = 2$ term (and higher-order terms) renders the new equation quite complicated. It will be noted, however, that for this case (moderate ε_2) the original breakage equation can be handled easily with existing methods (series solution (5) or numerically [30]). The picture is quite different as ε_2 becomes small. In this case the solution of the original equation becomes more difficult. However, in the new equation it suffices to retain only the first term of the expansion, that has a very simple form. It is interesting that the leading-order mass loss term (23a) is equivalent to that of erosion-breakage with fragments of fixed sizes and breakage rate $xb(x)$. This similarity does not hold for higher-order terms as one can ascertain by examining the $j = 2$ term (23b), which is quite different for the two processes [19]. Finally the erosive breakage equation for a homogeneous kernel takes the form

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial [g x b(x) f(x,t)]}{\partial x} + \int_{x/\varepsilon_1}^{\infty} \frac{1}{y} P_1(x/y) b(y) f(y,t)\, \mathrm{d}y \tag{24}$$

where $g = n_1 \varepsilon_2$ is the volume fraction that a parent particle loses per breakage event and the product $gxb(x)$ is the mass erosion rate.

3. Methods of solution

3.1. Decomposition to generations—analytical solution

By inspection of the series solution (5a) of the original breakage equation one can infer that the ith term of the series represents the size distribution of the particles that have undergone breakage $i+1$ times. The additional term is simply the size distribution of the remaining (unbroken) initial particles. For breakage kernels which are not close to the uniform one, the various terms of the series can be widely differing functions, resulting in a multimodal final distribution. For this reason it would be very useful to decompose the original equation to a hierarchy of equations that must be solved for the modes of the distribution. This approach has been exploited by Liou *et al* [31] for the solution of the growth–breakage population balance describing the dynamics of microbial and cell cultures. The potential of the approach of Liou *et al* to overcome problems encountered in the conventional solution methods of the population balance equation arising in biotechnology has been stressed by Villadsen [32]. In [31] the modes of the distribution are called *generations* (this term is used henceforth) and the development of the method is rather intuitive and based on biological system considerations and not on mathematical arguments as in this paper. Furthermore, the direct application of the method of generations to equation (4) for an erosion kernel is useless because an enormous number of generations is needed to describe the evolution of the eroded particles while only a few generations are needed for the fragments. This leads to a reformulation of the method of generations so that after a breakage event only the small fragments are assigned a new generation index whereas the large fragment remains in the same generation. This approach is equivalent to applying the method of generations directly to equation (24) instead of (4). For convenience the index $i = 1$ represents the eroded initial particles, and the index $i > 1$ particles which are the fine fragments of a cascade of $i-1$ breakage events. In what follows the term *parent particles* is used for the generation with $i = 1$ and the term *fragments* is used for the other generations.

The hierarchy of equations for the size distribution of the generations is

$$\frac{\partial f_1(x,t)}{\partial t} = \frac{\partial[gxb(x)f_1(x,t)]}{\partial x} \tag{25a}$$

$$\frac{\partial f_i(x,t)}{\partial t} = \frac{\partial[gxb(x)f_i(x,t)]}{\partial x} + \int_{x/\varepsilon_1}^{\infty} \frac{1}{y} P_1(x/y)b(y)f_{i-1}(y,t)\,\mathrm{d}y \quad \text{for} \quad i = 2,\ldots,N-1 \tag{25b}$$

$$\frac{\partial f_N(x,t)}{\partial t} = \int_{x/\varepsilon_1}^{\infty} \frac{1}{y} P_1(x/y)b(y)f_{N-1}(y,t)\,\mathrm{d}y \tag{25c}$$

where it has been assumed that no breakage occurs for particles of the Nth generation. The above equations must be solved successively with initial conditions $f_1(x,0) = f_0(x)$ and $f_i(x,0) = 0$ for $i > 1$. The complete size distribution is obtained by a simple superposition of the size distributions of the generations, as

$$f(x,t) = \sum_{i=1}^{N} f_i(x,t). \tag{26}$$

The method of generations is considered useful for the general breakage problem, but it seems to be perfectly suited to the present case of erosive breakage. In the latter, the solution displays important features in several size scales so that an appropriate discretization for a successful numerical treatment seems unfeasible. On the other hand the solutions of the generations equations are well behaved functions amenable to conventional treatment. A particular advantage of the generation equations is that they can be solved analytically using the method of characteristics [33]. Taking into consideration the general solution of the growth population balance given by Williams [34] the following result is obtained:

$$f_1(x,t) = \frac{1}{xb(x)} B^{-1}(B(x)+t)b[B^{-1}(B(x)+t)]f_0(B^{-1}(B(x)+t)) \tag{27a}$$

$$f_i(x,t) = \frac{1}{xb(x)} \int_0^t x(t')b(x(t')) \int_{x(t')/\varepsilon_1}^{\infty} \frac{1}{y} P_1(x(t')/y)b(y) f_{i-1}(y,t')\,\mathrm{d}y\,\mathrm{d}t' \quad \text{for} \quad i = 2, \ldots, N-1 \tag{27b}$$

$$f_N(x,t) = \int_0^t \int_{x/\varepsilon_1}^{\infty} \frac{1}{y} P_1(x/y)b(y) f_{N-1}(y,t')\,\mathrm{d}y\,\mathrm{d}t' \tag{27c}$$

where $B(x) = \int \frac{1}{gxb(x)}\,\mathrm{d}x$, $B^{-1}(z)$ is the solution to equation $z = B(x)$ and $x(t') = B^{-1}(t - t' + B(x))$.

3.2. Moments method

Although the above analytical solution can be used for practical purposes [31], it is rather complicated. On the other hand, frequently the gross features and not the details of the distribution are of interest. This is the reason for extensively using the method of moments for solving population balances [35]. Additionally, the method of moments has been used in the physics literature to obtain a physical insight from the structure of the solution [36]. In order to proceed with the method of moments and obtain analytical solutions the breakage rate is assumed to be of the power-law form $b(x) = x^{\nu}$.

The kth moment of the ith generation is defined as

$$M_{i,k} = \int_0^{\infty} x^k f_i(x,t)\,\mathrm{d}x. \tag{28}$$

The set of equations (25) is multiplied by x^k and integrated with respect to x from 0 to ∞. The product rule is used for integrating the term with the derivative together with the fact that $f(\infty, t) = 0$ due to physical considerations. As regards the double integral arising from the integral term, the order of integration is interchanged and the inner variable is scaled appropriately to permit the separation of the two integrals according to a standard procedure in the breakage literature. The resulting system, that describes the evolution of moments, is ($i = 1, 2, \ldots, N$, $k = 0, 1, 2, \ldots, \infty$)

$$\frac{\mathrm{d}M_{i,k}}{\mathrm{d}t} = -(1 - \delta_{iN})gkM_{i,k+\nu} + (1 - \delta_{i1})J_k M_{i-1,k+\nu} \tag{29}$$

where the Kronecker delta (δ_{ij} is equal to 1 for $i = j$ and equal to 0 for $i \neq j$) is used to obtain a system in compact form, and

$$J_i = \int_0^{\varepsilon_1} z^i P_1(z)\,\mathrm{d}z. \tag{30}$$

This quantity is of the order of ε_1^i and it is in general different from the quantity $n_i \varepsilon_1^i$, except for the case $i = 1$ where $J_1 = g$, and for the binary kernel where the two quantities are equal for every value of i. The above system is solved with initial conditions $M_{1,k}(0) = M_{k0}$ (the moments of the initial size distribution) and $M_{i,k}(0) = 0$ for $i > 1$.

For the case of constant breakage rate (exponent $\nu = 0$) the system (29) is closed and admits the following analytical solution:

$$M_{i,k} = M_{k0} J_k^{i-1} \mathrm{e}^{-gkt} \frac{t^{i-1}}{(i-1)!} \qquad \text{for} \quad i = 1, \ldots, N-1 \tag{30a}$$

$$M_{N,k} = M_{k0} J_k^{N-1} \frac{1}{(N-2)!} \left[\frac{(N-2)!}{(gk)^{N-1}} - \frac{\mathrm{e}^{-gkt}}{gk} \sum_{i=0}^{N-2} \frac{(N-2)!\,t^{N-2-i}}{(gk)^i (N-2-i)!} \right]. \tag{30b}$$

In the more general case of arbitrary ν the system is not closed with more unknowns than equations. A correlation between the moments (that is, an assumption for the particular shape of the distribution) is needed in order to close the system. The usually assumed shapes are the log-normal distribution for coagulation or/and growth equations [37] and the gamma distribution for the breakage equation [38]. Although the original equation is of the breakage type, the derived erosion equation for a particular generation, which is to be solved with the moments method, is of the growth type; thus, the assumption of the log-normal distribution is the preferred choice. More specifically, it is assumed that the size distribution of the ith generation has the form

$$f_i(x,t) = \frac{N_i}{\sqrt{2\pi\sigma_i}}\frac{1}{x}\exp\left[-\frac{1}{2\sigma_i}\ln^2\left(\frac{x}{\bar{x}_i}\right)\right] \tag{31}$$

where $N_i = M_{i,0}$ is the dimensionless number concentration, σ_i is the dispersivity defined as $\sigma_i = \ln(\frac{M_{i,2}M_{i,0}}{M_{i,1}^2})$ and $\bar{x}_i$ is the logarithmic mean size of the distribution. The mass fraction $\varphi_i = M_{i,1}$ is related to the above quantities through

$$\varphi_i = N_i \bar{x}_i \exp\left(\frac{\sigma_i}{2}\right). \tag{32}$$

After some algebra the following system for the evolution of the quantities N_i, φ_i, σ_i is derived:

$$\frac{\mathrm{d}N_i}{\mathrm{d}t} = (1-\delta_{i1})J_0 N_{i-1}^{1-\nu}\varphi_{i-1}^{\nu}\exp\left(\frac{\nu^2-\nu}{2}\sigma_{i-1}\right) \tag{33a}$$

$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}t} = -(1-\delta_{iN})g\frac{\varphi_i^{\nu+1}}{N_i^{\nu}}\exp\left(\frac{\nu^2+\nu}{2}\sigma_i\right) + (1-\delta_{i1})g\frac{\varphi_{i-1}^{\nu+1}}{N_{i-1}^{\nu}}\exp\left(\frac{\nu^2+\nu}{2}\sigma_{i-1}\right) \tag{33b}$$

$$\begin{aligned}\frac{\mathrm{d}\sigma_i}{\mathrm{d}t} &= (1-\delta_{i1})J_0\frac{\varphi_{i-1}^{\nu}}{N_i N_{i-1}^{\nu-1}}\exp\left(\frac{\nu^2-\nu}{2}\sigma_{i-1}\right) + (1-\delta_{iN})2g\frac{\varphi_i^{\nu}}{N_i^{\nu}}\exp\left(\frac{\nu^2+\nu}{2}\sigma_i\right)\\ &\quad -(1-\delta_{i1})2g\frac{\varphi_{i-1}^{\nu+1}}{\varphi_i N_{i-1}^{\nu}}\exp\left(\frac{\nu^2+\nu}{2}\sigma_{i-1}\right) - (1-\delta_{iN})2g\frac{\varphi_i^{\nu}}{N_i^{\nu}}\exp\left(\frac{\nu^2+3\nu}{2}\sigma_i\right)\\ &\quad +(1-\delta_{i1})J_2\frac{N_i\varphi_{i-1}^{\nu+2}}{\varphi_i^2 N_{i-1}^{\nu-1}}\exp\left(\frac{(\nu+2)(\nu+1)}{2}\sigma_{i-1}-\sigma_i\right).\end{aligned} \tag{33c}$$

The above system of ODEs has to be solved for $i = 1, 2, \ldots, N$ with initial conditions $N_1 = \varphi_1 = 1$, $\sigma_1 = \sigma_0$ and $N_i = \varphi_i = \sigma_i = 0$ for $i > 1$. Special care must be taken in the numerical integration of this system because for $i > 1$ it is singular at $t = 0$. Thus, the integration must be initialized using an equivalent non-singular system including only the source terms (associated with the $i-1$ generation). To continue, at a small but finite value of time, this simplified system is replaced by the complete one (equations (33)). The advantage of using the dispersivity σ instead of M_2 is that σ is of order 1 whereas M_{i2} tends to sharply decrease (by orders of magnitude) as the generation index i increases.

The above method with a log-normal distribution can be extended to include more than three moments via a generalized correlation between moments [39]. Also the restriction of the power form for $b(x)$ can be relaxed via the quadrature method of moments [40]. Overall, the log-normal approximation is considered the best compromise between reduced computational effort and adequate level of accuracy to obtain the salient features of the evolving distributions. It is worth noting that the solution of the breakage equation with constant breakage rate takes asymptotically the log-normal form [41, 42]. Another interesting observation is that the multimodal log-normal method used here for the solution of the erosive breakage equation is a formal equivalent of bimodal [43] and trimodal [44] log-normal methods of moments that have been used extensively for solving the general dynamic equation of aerosols.

4. Case study

4.1. Binary erosion-breakage kernels

The simplest erosion kernels will be described next. These are binary kernels such that $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and $P_1(z) = P_2(1-z)$; the simplest is the monodisperse erosion kernel $P_1(z) = \delta(z-\varepsilon)$ for which $J_i = \varepsilon^i$. The uniform erosion kernel is $P_1(z) = 1/\varepsilon$ with $J_i = \varepsilon^i/(i+1)$. The power-law erosion kernel that assigns greater probability to smaller fragments has the form $P_1(z) = (n+1)/\varepsilon(1-z/\varepsilon)^n$ with $J_i = B(i+1, n+1)$, where $B(x, y)$ is the beta function. Obviously the uniform kernel is a member of the family of the power kernels with $n = 0$.

4.2. Constant-breakage-rate case

The particular case of constant breakage rate ($\nu = 0$) is studied here in detail because it is amenable to a simplified treatment and its solution exhibits a very interesting behaviour. Using equation (27*a*) for $\nu = 0$ results in

$$f_1(x, t) = e^{gt} f_0(x e^{gt}) \tag{34}$$

irrespective of the type of breakage kernel. The monodisperse erosion-breakage kernel admits an analytical solution as follows. Substituting the monodisperse kernel in equation (27*b*) and using the characteristic $x(t') = x e^{-g(t-t')}$ results in the recursive relation ($i = 2, \ldots, N-1$):

$$f_i(x, t) = \int_0^t f_{i-1}\left(\frac{x e^{g(t-t')}}{\varepsilon}, t'\right) e^{g(t-t')}\, dt'. \tag{35}$$

Using f_1 to start with, the recursive integration can be performed analytically to obtain

$$f_i(x, t) = \frac{e^{\varepsilon t}}{\varepsilon^{i-1}} \frac{t^{i-1}}{(i-1)!} f_0\left(\frac{e^{\varepsilon t}}{\varepsilon^{i-1}}\right). \tag{36}$$

From the first two moments of the above distributions (or alternatively using equation (30*a*)) one obtains

$$M_{i,0} = \frac{t^{i-1}}{(i-1)!} \tag{37a}$$

$$M_{i,1} = \frac{t^{i-1}}{(i-1)!} \varepsilon^{i-1} e^{-\varepsilon t}. \tag{37b}$$

The well known self-similarity transformation, originally proposed by Friedlander [45] for the solution of the coagulation equation, is here defined at the generation level as

$$\bar{f}_i(\bar{x}) = \frac{M_{i,1}}{M_{i,0}^2} f\left(\frac{x M_{i,0}}{M_{i,1}}\right). \tag{38}$$

Substitution into equation (36) gives $\bar{f}_i(\bar{x}) = f_0(\bar{x})$, which means that the shape of the size distribution for each generation is similar to that of the initial distribution. This type of self-similarity is quite different from the conventional one since it does not hold only in the large-time limit but for all times, and it is not independent of the initial distribution, but it has exactly its shape. The present type of self-similarity has also been noticed in the study of the (equivalent to the $i = 1$ equation of the present problem) polymer degradation with chain-end scission [19], a case for which it has been argued that no self-similarity solution exists [46]. The above (generation-level) self-similarity solution is restricted to constant breakage rate ($\nu = 0$) and monodisperse erosion kernel. Furthermore, it should not be confused with the

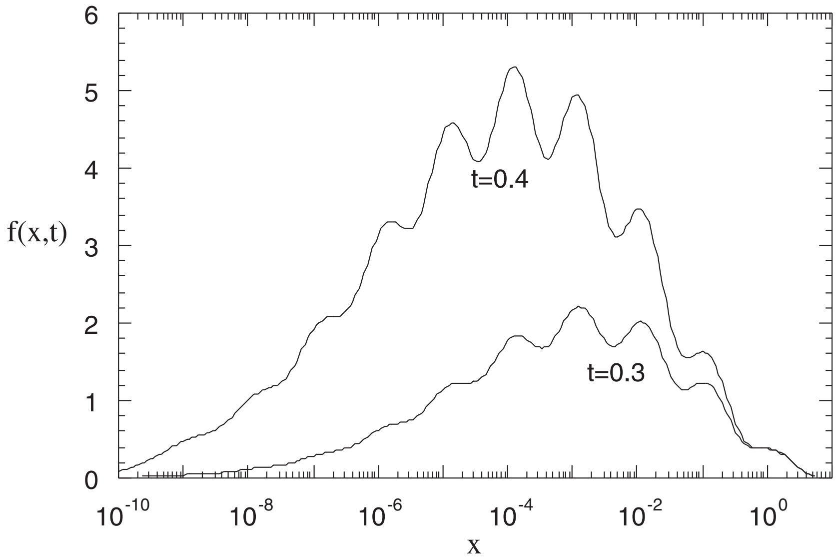

Figure 1. Particle size distribution for constant breakage rate and monodisperse binary erosion breakage kernel ($\varepsilon = 0.1$) for two values of breakage time. Initial condition $f_0(x) = x\mathrm{e}^{-x}$.

global asymptotic self-similarity solution which exists for every homogeneous breakage kernel, including the homogeneous erosion kernel studied here.

An interesting consequence of equation (36) is that, in the case of a monodisperse initial distribution, the size distribution of each generation is monodisperse as well. This is due to the fact that, as the particles of generation i 'move' to smaller sizes, the particles of generation $i-1$ 'move' in such a way that (at every instance) the fragments they produce have the current size of the particles of generation i. Summarizing, the solution of equation (24) for constant breakage rate and monodisperse binary erosion kernel is

$$f(x,t) = \mathrm{e}^{\varepsilon t} \sum_{i=1}^{\infty} \frac{1}{\varepsilon^{i-1}} \frac{t^{i-1}}{(i-1)!} f_0\left(\frac{\mathrm{e}^{\varepsilon t}}{\varepsilon^{i-1}}\right). \tag{39}$$

The evolving particle size distribution (based on the above equation) for an initial distribution $f_0(x) = x\mathrm{e}^{-x}$ is shown in figure 1 for $\varepsilon = 0.1$ and two values of time ($t = 0.3$ and 0.4). Figure 2 is similar to figure 1 but with the narrower initial distribution $f_0(x) = (256/6)x^3\mathrm{e}^{-4x}$. In both cases the multimodality of the fragment distribution is evident. However, with broader initial distributions overlapping between the generations increases.

For a more general kernel than the binary monodisperse one there is no analytical solution, although some features of the solution can be obtained using the moments equation (30). Thus, the mean size of generation i is found to be $x_{mi} = z_m^{i-1}\mathrm{e}^{-gt}$, where $z_m = \frac{g}{v-1}$ is the mean size of the fragment size distribution. The dispersivity of generation i is $\sigma = \sigma_0 + (i-1)\sigma_{\text{fragment}}$, where $\sigma_{\text{fragment}} = \ln(\frac{(v-1)J_2}{g})$, may be called the dispersivity of the fragment size distribution and v is defined in equation (9). This means that the dispersivity of each generation tends to increase, compared with the dispersivity of the preceding one, by an amount equal to the dispersivity of the fragment size distribution. For the monodisperse kernel $\sigma_{\text{fragment}} = 0$; thus, all the generations have the same dispersivity σ_0 as the initial distribution (already shown). As a final comment for the constant-breakage-rate ($\nu = 0$) case, it is noted that by substituting $\nu = 0$ the approximate system for the moments (33) degenerates to the exact one (29), which is closed and the approximation procedure is unnecessary.

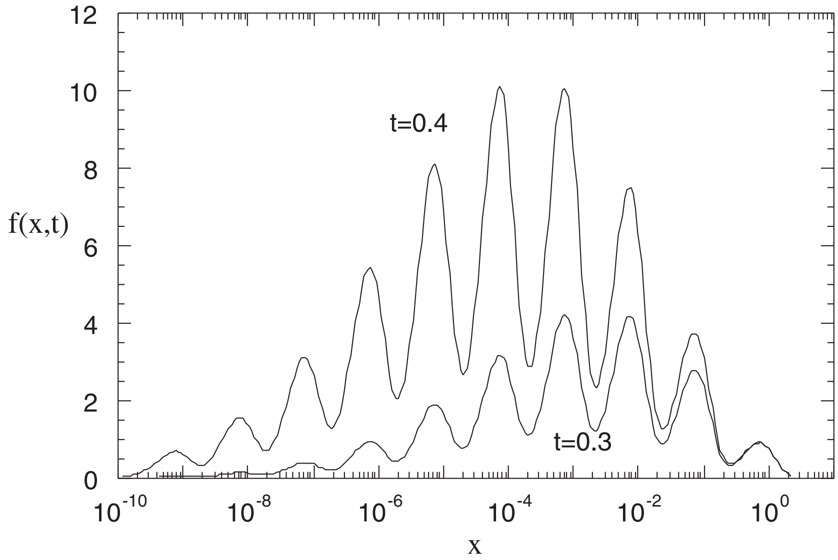

Figure 2. Particle size distribution for constant breakage rate and monodisperse binary erosion breakage kernel ($\varepsilon = 0.1$) for two values of breakage time. Initial condition $f_0(x) = (256/6)x^3 e^{-4x}$.

4.3. *Study of the $i = 2$ generation*

In order to examine the influence of the breakage rate exponent ν on the fragment size distribution, the system (33) is solved numerically, with a fourth-order Runge–Kutta integrator with self-adaptive step and prespecified accuracy [47], for several values of ν, monodisperse binary erosion kernel ($\varepsilon = 0.1$) and monodisperse initial distribution. The evolution of the mass fraction for the $i = 2$ generation is shown in figure 3. As ν increases the evolution becomes slower but the maximum mass fraction φ_2 increases. This is due to the 'sharpening' of the dependence of the breakage rate on the particle size. Here 'sharp' dependence means decreasing rate of mass loss with decreasing size, which results in mass accumulation in the generation. The evolution of the dispersivities which correspond to the mass fractions of figure 3 is shown in figure 4. The dispersivity initially increases and after reaching a maximum decreases. For large values of ν the maximum dispersivity appears at large values of the mass fraction φ_2 but as ν decreases it occurs at negligible φ_2. In general the absolute value of the dispersivity remains small (maximum $= 0.15$), keeping in mind that for $\nu = 0$ σ_2 is identically zero.

4.4. *Study of the $N = 2$ case*

To obtain an insight into the structure of the solution some simple cases will be examined, for $N = 2$, power breakage rate and monodisperse erosion kernel. The general solution for f_1 (irrespective of N and erosion kernel) with power breakage rate is

$$f_1(x,t) = \frac{(x^{-\nu} - g\nu t)^{-(\nu+1)/\nu}}{x^{\nu+1}} f_0[(x^{-\nu} - g\nu t)^{-1/\nu}] \qquad \text{for} \quad \nu \neq 1 \tag{40}$$

where $x^{\nu} < \frac{1}{g\nu t}$. It is noted that if this condition is not satisfied $f_1(x,t) = 0$. This discontinuity follows from the fact that because of particle erosion the maximum particle size at time t is

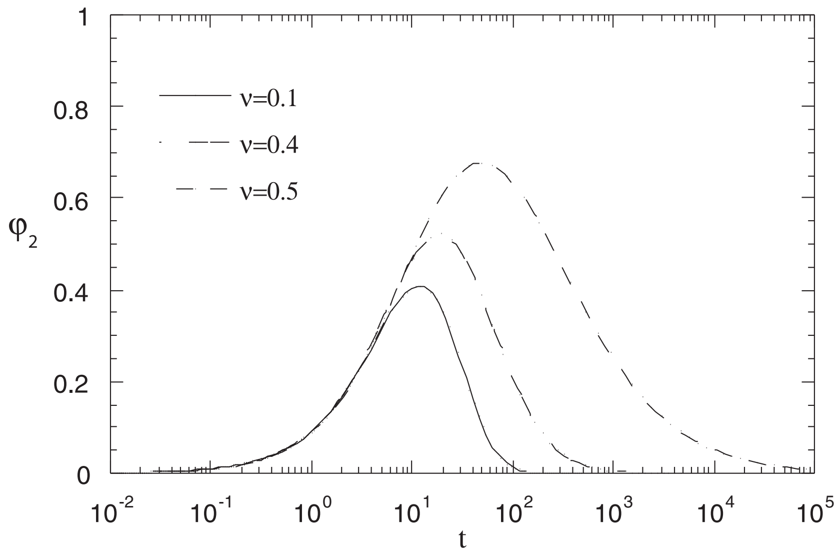

Figure 3. Evolution of the mass fraction φ_2 of the second generation of particles undergoing breakage, under conditions of monodisperse binary erosion kernel ($\varepsilon = 0.1$), for several values of the breakage rate exponent ν.

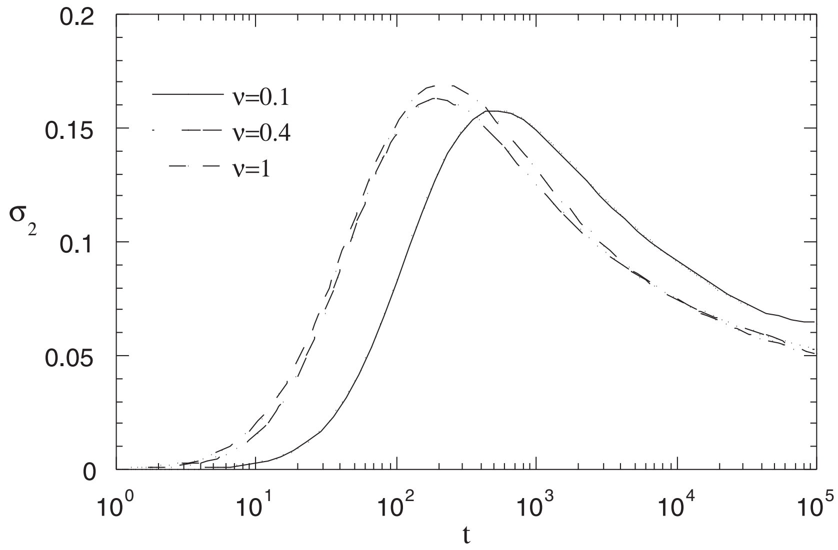

Figure 4. Evolution of the dispersivity σ_2 of the second generation of particles undergoing breakage, under conditions of monodisperse binary erosion kernel ($\varepsilon = 0.1$) for several values of the breakage rate exponent ν.

$x = (\frac{1}{g\nu t})^{1/\nu}$. For the limiting case $\nu = 0$ the result has already been given in the analysis of the constant-breakage-rate case. For $N = 2$ and monodisperse breakage kernel, equation (27*c*) takes the form

$$f_2(x, t) = \int_0^t b(x/\varepsilon) f_1(x/\varepsilon, t')\, \mathrm{d}t'. \tag{41}$$

Substituting f_1 from equation (40) and using a new variable of integration results in

$$f_2(x,t) = \frac{1}{x}\int_{x/\varepsilon}^{((x/\varepsilon)^{-\nu}-\varepsilon\nu t)^{-1/\nu}} f_0(z)\,\mathrm{d}z$$

or (42)

$$f_2(x,t) = \frac{1}{x}\int_{x/\varepsilon}^{x\mathrm{e}^{\varepsilon t}/\varepsilon} f_0(z)\,\mathrm{d}z \qquad \text{for} \quad \nu = 1.$$

4.5. *Erosion breakage with size cut-off*

In many practical cases there is a critical size x_c for the parent particles; i.e. particles of size smaller than x_c (referred to as the *cut-off* size) do not suffer breakage. For example, in the turbulent flow of liquid–liquid dispersions the flow field cannot cause breakage of droplets below a certain size related to the turbulent eddy structure [48]. This behaviour appears to be general in nature since every physical system has a characteristic cut-off size, even if it is only at the quantum level. The breakage problem with a size cut-off has been formulated and solved for power breakage rate and kernel by Huang *et al* [1]. The existence of the critical size x_c has the consequence that after sufficient time a *static* steady-state distribution is established because all particles are smaller than x_c. If x_c is in between the smaller parent particle and the larger fragment (i.e. $f_0(x) = 0$ for $x > x_c/\varepsilon_1$) then the above problem is simply described by equation (25*a*) for $x > x_c$ and by equation (25*c*) for $N = 2$ and $x < x_c$. The steady-state distribution can be obtained directly, avoiding any transient results, by a modification of the method developed by Kostoglou and Karabelas [49] for general breakage. As regards the fate of parent particles, it can be easily shown that in the steady state they are accumulated in the region between x_c and $x_c/(1-\varepsilon_2)$. To obtain the steady-state distribution of the fragments, the following function is introduced:

$$L(x) = \int_0^\infty b(x) f_1(x,t)\,\mathrm{d}t. \tag{43}$$

The two equations of this problem are integrated with respect to x from 0 to ∞, leading to a new time-independent problem:

$$-f_0(x) = \frac{\partial gxL(x)}{\partial x} \qquad \text{for} \quad x > x_c \tag{44}$$

$$f_2(x,\infty) = \int_{x/\varepsilon_1}^{x_c/\varepsilon_1} \frac{1}{y} P_1(x/y) L(y)\,\mathrm{d}y \qquad \text{for} \quad x < x_c. \tag{45}$$

Solving (44) for $L(x)$ and substituting in (45) the following result is obtained:

$$f_2(x,\infty) = \int_{x/\varepsilon_1}^{(x_c/\varepsilon_1)} \int_y^{(x_c/\varepsilon_1)} \frac{1}{gy^2} P_1(x/y) f_0(z)\,\mathrm{d}y\,\mathrm{d}z \qquad \text{for} \quad x < x_c. \tag{46}$$

5. Concluding remarks

A general study of the breakage problem with a homogeneous kernel of erosion type is attempted in this paper. The use of the conventional breakage equation to solve a problem of this type is inefficient because of the disparity of scales introduced by the size difference between the fragments. For improvement, a new equation (erosion-breakage equation) is derived as a first-order perturbation expansion of the original problem, with respect to the fractional mass reduction of the parent particle per breakage event. However, handling the new equation is still difficult because its solution is multimodal; thus, it is decomposed into a system of equations for the modes using a method of generations. This system of equations is solved analytically with the method of characteristics. Additionally, an approximate method of solution is demonstrated, based on the methods of moments with log-normal distribution. Using the above methods of solution, explicit results are presented for several simple cases. These results reveal features which would be impossible to obtain by a numerical solution of the conventional breakage equation. Of special interest is the existence (for a particular case) of self-similarity solutions at the level of generation, in addition to the well known global self-similarity solution of the homogeneous breakage equation. It is believed that the methodology developed in this paper has definite advantages, over any other available method, for analysing processes where cascade fractional breakage occurs, with the size of the larger fragment close to that of the parent particle.

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